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On steady flow of a reactive variable viscosity fluid in a cylindrical pipe with an isothermal wall

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Abstract

Purpose – This paper aims to demonstrate an application of a novel approximation technique, for a function whose partial series is known, to a problem in thermal ignition in a combustible variable viscosity fluid.

Design/methodology/approach – Analytical solutions are constructed for the governing non-linear boundary-value problem using regular perturbation technique (RPT) coupled with computer-extended series solution (CESS) and a special type of Hermite-Padé approximant.

Findings – The steady state thermal ignition criticality conditions and their dependent on both Frank-Kamenetskii and viscous heating parameters are accurately obtained. The results also revealed the rapid convergence of the approximation procedure with gradual increase in the number of series coefficients utilized in the approximants.

Originality/value – The analytical and computational procedures utilized in this paper are advocated as an effective tool for investigating several other parameter dependent nonlinear boundary-value problems.

Keywords Pipes, Flow, Viscosity; Thermal conductivity

Paper type Research paper

1. Introduction

In petrochemical industries and petroleum refineries, studies related to thermal ignition criticality and heat transfer in a reactive variable viscosity fluid are extremely useful in order to ensure safety of life and properties (Bebernes and Eberly, 1989; Bowes, 1984). Thermal ignition occurs when a reaction produces heat too rapidly for a stable balance between heat production and heat loss to be preserved. Hence, it is important to know the critical values of the basic physical quantities, such as the ambient temperature, surface characteristics, the chemistry of the reacting material and the physical geometry at which thermal ignition occurs (Balakrishnan *et al.*, 1996; Bebernes and Eberly, 1989; Makinde, 2004, 2005a). Therefore, the concept of thermal criticality or non-existence of steady-state solution of non-linear reaction diffusion problems for certain parameter values is very important from the application point of view. This characterizes the thermal stability properties of the material under consideration and the onset of thermal runaway phenomenon.

The classical formulation of this type of problem was first introduced by Frank Kamenetskii (1969) as shown in Figure 1. Neglecting the reacting viscous

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incompressible fluid consumption, the equations for the heat balance and momentum in the original variables together with the boundary conditions can be written as:

$$\frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\bar{r}\frac{\mathrm{d}T}{\mathrm{d}\bar{r}}\right)+\frac{QC_0A}{k}\mathrm{e}^{-(E/RT)}+\frac{\mu}{k}\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{r}}\right)^2=0,\frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left(\mu\bar{r}\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{r}}\right)=-G, \tag{1}$$

$$u=0, \quad T=T_0, \quad \text{on } \bar{r}=a, \tag{2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}\bar{r}}=\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{r}}=0, \quad \text{on } \bar{r}=0, \tag{3}$$

where T is the absolute temperature, $\bar{u}$ the fluid axial velocity, G the constant axial pressure gradient, T_0 the wall reference temperature, k the thermal conductivity of the material, Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the pipe characteristic radius, $(\bar{z},\bar{r})$ the distance measured in the axial and radial directions, respectively. It is assumed that the dynamic viscosity of the reactive viscous fluid under investigation is temperature dependent, that is:

$$\mu=\mu_0\mathrm{e}^{E/RT}, \tag{4}$$

where μ_0 is the fluid reference viscosity. The following dimensionless variables are introduced into equation (1):

$$\begin{aligned}\theta&=\frac{E(T-T_0)}{RT_0^2},\\ \varepsilon&=\frac{RT_0}{E},\\ y&=\frac{\bar{r}}{a},\\ \lambda&=\frac{QEAa^2C_0\mathrm{e}^{-E/RT_0}}{T_0^2Rk},\\ W&=\frac{\mu_0\bar{u}\mathrm{e}^{E/RT_0}}{Ga^2},\\ \beta&=\frac{G^2a^2}{4QC_0A\mu_0},\end{aligned} \tag{5}$$

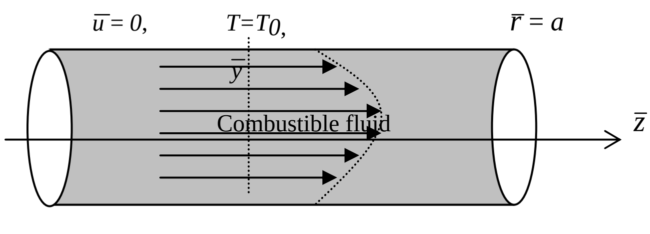


Figure 1.
Geometry of the problem

and obtain the dimensionless governing equation together with the corresponding boundary conditions as

$$\frac{dW}{dr} = -\frac{r}{2}e^{(\theta/(1+\varepsilon\theta))}, \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \lambda(1+\beta r^2)e^{(\theta/(1+\varepsilon\theta))} = 0, \tag{6}$$

$$\frac{d\theta}{dr}(0) = 0, \quad \theta(1) = W(1) = 0, \tag{7}$$

where $\lambda, \varepsilon, \beta,$ represent the Frank-Kamenetskii parameter, activation energy parameter and the viscous heating parameter, respectively. In the following sections, equations (6) and (7) are solved using both regular perturbation and multivariate series summation techniques (Guttamann, 1989; Hunter and Baker, 1979; Makinde, 2004, 2005a, b; Makinde and Osalusi, 2005; Sergeyev and Goodson, 1998; Tourigny and Drazin, 2000).

2. Method of solution

The non-linear nature of the equations (6) and (7) precludes its solution exactly; hence, we employed a regular perturbation technique (RPT) in order to obtain an approximate solution of the problem. It is convenient to take a power series expansion in the Frank-Kamenetskii parameter λ, that is $\theta = \sum_{i=0}^{\infty}\theta_i\lambda^i$. Substituting the solution series into equations (6) and (7) and collecting the coefficients of like powers of λ, we obtained and solved the equations of the coefficients of solution series iteratively. The solution for the temperature and velocity fields are given as:

$$\begin{aligned}\theta(r) = &-\frac{\lambda(r^2-1)}{16}(\beta r^2+4+\beta) + \frac{\lambda^2(r^2-1)}{9,216}(9r^6\beta^2+80\beta r^4+9\beta^2r^4 \\ &-27\beta^2r^2+144r^2-64\beta r^2-208\beta-432-27\beta^2) + O(\lambda^3)\end{aligned} \tag{8}$$

$$\begin{aligned}W(r) = &-\frac{1}{4}(r^2-1) + \frac{\lambda}{192}(r^2-1)^2(\beta r^2+2\beta+6) \\ &-\frac{\lambda^2}{184,320}(r^2-1)^2(27r^6\beta^2+280\beta r^4+54\beta^2r^4+80\beta r^2-39\beta^2r^2 \\ &+720r^2-132\beta^2-840\beta-1,440) + O(\lambda^3)\end{aligned} \tag{9}$$

Using computer symbolic algebra package (MAPLE), we obtained the computer extended series solution (CESS) up to the first 22 terms in equations (8) and (9) as well as the series for the fluid maximum temperature $\theta_{max} = \theta(r = 0; \lambda, \varepsilon, \beta)$.

3. Hermite-Padé approximation technique

The main tool of this paper is a simple technique of series summation based on the generalization of Padé approximants and may be described as follows. Suppose that the partial sum:

$$U_{N-1}(\lambda) = \sum_{i=0}^{N-1} a_i\lambda^i = U(\lambda) + O(\lambda^N) \quad \text{as} \quad \lambda \to 0, \tag{10}$$

is given. We are concerned with the bifurcation study by analytic continuation as well as the dominant behaviour of the solution by using partial sum in equation (10).

We expect that the accuracy of the critical parameter (λ_c) will ensure the accuracy of the solution. It is well known that the dominant behaviour of a solution of a differential equation can often be written as Guttamann (1989):

$$U(\lambda) \approx \begin{cases} K(\lambda_c - \lambda)^\alpha & \text{for } \alpha \neq 0, 1, 2, \ldots \\ K(\lambda_c - \lambda)^\alpha \ln|\lambda_c - \lambda| & \text{for } \alpha = 0, 1, 2, \ldots \end{cases} \quad \text{as } \lambda \to \lambda_c \tag{11}$$

where K is some constant and λ_c is the critical point with the exponent α. We shall make a simple hypothesis in the contest of nonlinear problems by assuming that $U(\lambda)$ is the local representation of an algebraic function of λ. Therefore, we seek an expression of the form:

$$F_d(\lambda, U_{N-1}) = A_{0N}(\lambda) + A_{1N}^d(\lambda)U^{(1)} + A_{2N}^d(\lambda)U^{(2)} + A_{3N}^d(\lambda)U^{(3)}, \tag{12}$$

such that:

$$A_{0N}(\lambda) = 1, \quad A_{iN}(\lambda) = \sum_{j=1}^{d+i} b_{ij}\lambda^{j-1}, \tag{13}$$

and:

$$F_d(\lambda, U) = O(\lambda^{N+1}), \quad \text{as } \lambda \to 0, \tag{14}$$

where $d \geq 1$, $i = 1, 2, 3$. Equation (13) normalizes F_d and ensures that the order of series A_{iN} increases as i and d increase in value. There are $3(2 + d)$ undetermined coefficients b_{ij} in the equation (13) and the requirement in equation (14) reduces the problem to a system of N linear equations for the unknown coefficients of F_d. The entries of the underlying matrix depend only on the N given coefficients a_i. Henceforth, we shall take:

$$N = 3(2 + d), \tag{15}$$

so that the number of equations equals the number of unknowns. Equation (14) is a new special type of Hermite-Padé approximants. Both the algebraic and differential forms of approximant in equation (14) are considered. For instance, if we let:

$$U^{(1)} = U, \quad U^{(2)} = U^2, \quad U^{(3)} = U^3, \tag{16}$$

the resulting set of cubic algebraic Hermite-Padé approximants enable us to obtain other solution branches of the underlying problem in addition to the one represented by the original series. Similarly, if we let:

$$U^{(1)} = U, \quad U^{(2)} = DU, \quad U^{(3)} = D^2U, \tag{17}$$

in equation (14), where D is the differential operator given by $D = \mathrm{d}/\mathrm{d}\lambda$. This leads to a set of second order differential approximants. It is an extension of the integral approximants idea by Hunter and Baker (1979) and enables us to obtain the dominant singularity in the flow field, that is, by equating the coefficient $A_{3N}(\lambda)$ in the equation (14) to zero. Furthermore, it is important to note that the rationale for choosing the degree of the A_{iN} in equation (13) is based on the simple technique of singularity determination in second order linear ordinary differential equation with polynomial coefficients as well as the possibility of multiple solution branches for the nonlinear problem (Makinde, 2004). In practice, one usually finds that the dominant singularities are located at zeroes of the leading polynomial $A_{3N}^{(d)}$ coefficients of the second order linear ordinary differential equations. Hence, some of the zeroes of $A_{3N}^{(d)}$ may provide approximations of the singularities of the series U and we expect that the accuracy of the singularities will ensure the accuracy of the approximants.

The critical exponent α_N can easily be found by using Newton's polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains $\alpha_N = 1/2$. If we assume a singularity of algebraic type as in equation (11), then the exponent may be approximated by:

$$\alpha_N = 1 - \frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}. \tag{18}$$

For details on the above procedure, interested readers can see (Guttamann, 1989; Hunter and Baker, 1979; Makinde, 2004, 2005a, b; Makinde and Osalusi, 2005; Sergeyev and Goodson, 1998; Tourigny and Drazin, 2000). We apply this procedure on the first twenty-two terms of the solution series as shown in the following section.

4. Results and discussion

Since, the fluid is incompressible and viscous, the above mathematical analysis is very suitable for highly combustible reactive liquid. It is important to note that the viscous heating parameter (β) depends on the liquid viscosity and increasing positive values of β indicate a gradual decrease in the reactive combustible liquid viscosity. The analytical and computational procedures highlighted in Section 3 above are utilized and we obtained the results as shown in Tables I and II.

D	N	θ_{max}	λ_c	α_{cN}
1	9	1.38654059395057	1.99999999999999	0.49999999999
2	12	1.38629435020816	2.00000000000000	0.50000000000
3	15	1.38629436111989	2.00000000000000	0.50000000000
4	18	1.38629436111989	2.00000000000000	0.50000000000
5	21	1.38629436111989	2.00000000000000	0.50000000000

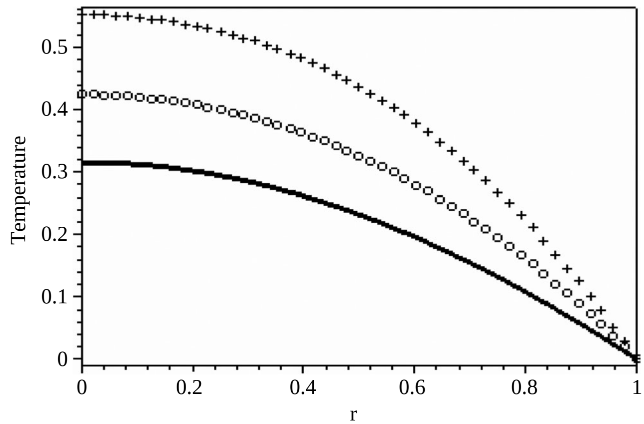
Table I. Computations showing the procedure rapid convergence for $\varepsilon = 0.0$, $\beta = 0.0$

β	θ_{max}	λ_c	α_{cN}
0	1.3862943611119	2.00000000000000	0.5000000
1	1.413378943801	1.62484015038031	0.5000000
2	1.426840289644	1.36327227049883	0.5000000
3	1.433109758511	1.17183579592218	0.5000000

Note: Computations showing thermal ignition criticality and maximum fluid temperature for various values of parameter (β), $\varepsilon = 0.0$

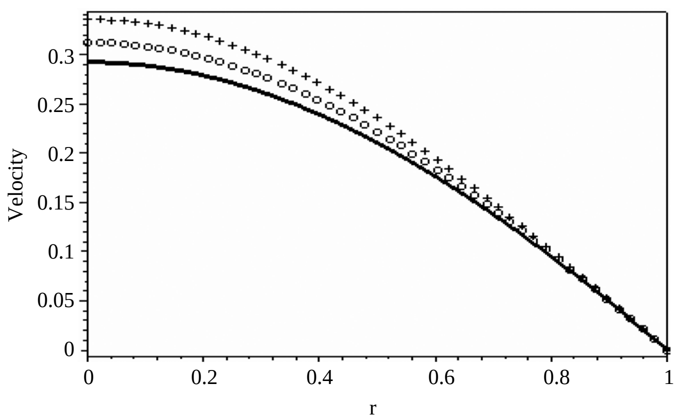
Table II.

Table I shows the rapid convergence of the dominant singularity λ_c (that is the thermal criticality conditions) together with its corresponding critical exponent α_c and maximum temperature for combustible variable viscosity fluid with gradual increase in the number of series coefficients utilized in the approximants. Table II shows the effect of viscous heating on the magnitude of thermal criticality for a reactive variable viscosity fluid at very large activation energy ($\varepsilon = 0$). It is noteworthy that a decrease in the magnitude of thermal criticality occurs due to a decrease in the combustible fluid viscosity (i.e. $\beta > 0$). Hence, viscous heating will enhance thermal ignition and steady flows of combustible fluid at low viscosity under Arrhenius kinetics will ignite faster than the one at high viscosity. Figures 2 and 3 show both the temperature and the velocity profiles. The fluid temperature increases with increasing values of viscous heating parameter. Similar profile is observed with fluid velocity, that is, a combustible fluid at low viscosity flows faster than the one at high viscosity. A slice of the bifurcation diagram for $\varepsilon = 0$ is shown in Figure 4. In particular, for every $\beta \geq 0$,



Notes: ________β=0; ooooooooβ=1; ++++++++β=2.

Figure 2.
Temperature profile for $\lambda = 0.5$; $\varepsilon = 0$



Notes: ________β=0; ooooooooβ=1; ++++++++β=2.

Figure 3.
Velocity profile for $\lambda = 0.5$; $\varepsilon = 0$

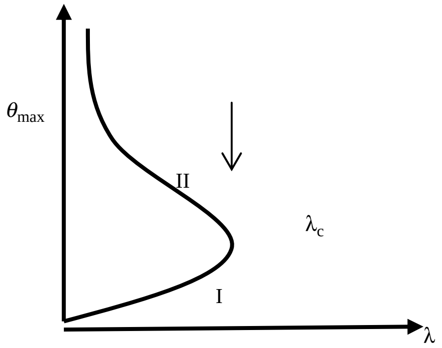


Figure 4.
A slice of approximate bifurcation diagram in the $(\lambda, \theta_{max}(\beta \geq 0, \varepsilon = 0))$ plane

there is a critical value λ_c (a turning point) such that, for $0 \leq \lambda < \lambda_c$ there are two solutions (labeled I and II) and the solution II diverges to infinity as $\lambda \rightarrow 0$.

5. Conclusions

The steady flow of a reactive variable viscosity fluid in a pipe with an isothermal wall is investigated using perturbation series summation and improvement techniques. A bifurcation study by analytic continuation of a power series in the bifurcation parameter for a particular solution branch is performed. The procedure reveals accurately the steady state thermal ignition criticality conditions as well as their dependent on both Frank-Kamenetskii and viscous heating parameters. Finally, the above analytical and computational procedures are advocated as effective tool for investigating several other parameter dependent nonlinear boundary-value problems.

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